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A CHARACTERIZATION OF INTEGRAL CURRENTS(1)

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1. **Introduction.** In [6] J. Král defined for each $r>0$ a function v_r^C which for $z\in \boldsymbol{R}^n$ and $r=\infty$ gives a measure of the generalized solid angle under which the boundary A of a bounded Borel set C is visible from z. For smooth A, $v_r^C(z)$ is the integral over the space of lines λ passing through z of the number of points in $A\cap\lambda\cap\{x : 0<|x-z|<r\}$. This function is used to give geometric conditions on an open set C which ensure the existence of solutions to the Neumann boundary value problem in C. In particular, Král shows that C has finite perimeter $\boldsymbol{P}(C)$ (a definition is in [6, 2.8]) if and only if $v_\infty^C(z)<\infty$ for $z\notin A$.

On the other hand, v_r^C is related to the formulas of integral geometry. We show in 4.5 that there exists an absolute constant γ such that if $\boldsymbol{P}(C)<\infty$, then

$$\int_{\boldsymbol{R}^n} v_r^C\, dH^n = \gamma r\boldsymbol{P}(C).$$

In the present paper we seek to extend the results of Král relating the geometry of the boundary of C with properties of v_∞^C to analogous propositions concerning boundaries of $k+1$ dimensional objects in $\boldsymbol{R}^n$, $0<k<n$. For smooth boundaries A the appropriate analogue of $v_\infty^C(z)$ is the integral $V^A(z)$ over the space of l planes B containing z, $k+l\geqq n$, of the $k+l-n$ dimensional Hausdorff measure of $A\cap B\sim\{z\}$. We use the slicing theory developed by H. Federer in [3] to extend V^A to the case where A is a k dimensional flat chain T. (If S is the n current obtained by integration over C, then ∂S is flat; $\boldsymbol{P}(C)$ equals the mass $\boldsymbol{M}(\partial S)$ of ∂S, and $V^{\partial S}=v_\infty^C$. Rectifiable currents are flat.)

In §5 we derive for V^T generalizations of the properties of v_∞^C obtained by Král in [6, §§1,2]. We also obtain an absolute continuity result for T which extends [5, 8.5].

In §6 we show that $\boldsymbol{M}(\partial S)<\infty$ if $V^{\partial S}(z)<\infty$ for sufficiently many points $z\in\boldsymbol{R}^n$. Combining this with results in §5 we obtain the extension of Král's characterization of sets of finite perimeter:

$\boldsymbol{M}(\partial S)<\infty$ if and only if $V^{\partial S}(z)<\infty$ for $z\notin \operatorname{spt}\partial S$. From this we obtain our characterization of integral currents by recalling from [5, 8.14] that if S is rectifiable and $\boldsymbol{M}(\partial S)<\infty$, then S is an integral current.

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Now suppose S is rectifiable and $\partial S = T + U$ with $M(T) < \infty$. In [3, 2.2] Federer proved that if the support of U has Hausdorff measure zero in dimension k, then S is an integral current, thus extending [5, 8.14]. We further extend this in §6 by showing that S is an integral current if for sufficiently many $z \in \boldsymbol{R}^n$,

$$(\operatorname{spt} U) \cap B \sim \{z\} = \varnothing$$

for almost all l planes B containing z.

Finally, we show in §7 that if T is obtained by integration over an oriented k dimensional submanifold T of class 1 and $z \in T$, then $V^T(z) < \infty$ for $k+l > n$, and also for $k+l = n$ if the tangent spaces of T satisfy a Hölder condition. (For $k = n-1$ such surfaces are the Lyapunov surfaces of classical potential theory.)

2. **Preliminaries.** The purpose of this section is to fix basic notation and terminology. We readopt the notation and terminology of [1] and [5].

Fix $n > 1$, and let $k < n$ and $l < n$ be positive integers such that $k + l \geqq n$. Denote

$$B = \boldsymbol{R}^n \cap \{x : x^1 = \cdots = x^{n-l} = 0\} \sim \{0\}$$

and $\psi_B = dx^{n-l+1} \wedge \cdots \wedge dx^n$, and orient B so that $B^{\rightarrow} = e_{n-l+1} \wedge \cdots \wedge e_n$.

The map

$$f\colon \mathrm{SO}(n) \times B \to \boldsymbol{R}^n \times \mathrm{SO}(n), \qquad f(g, b) = (g(b), g),$$

embeds $\mathrm{SO}(n) \times B$ as a proper submanifold Φ of $\boldsymbol{R}^n \times \mathrm{SO}(n)$ of dimension $m+l$, where $m = \frac{1}{2}n(n-1)$ is the dimension of $\mathrm{SO}(n)$. Defining

$$p\colon \boldsymbol{R}^n \times \mathrm{SO}(n) \to \boldsymbol{R}^n, \qquad p(x, g) = x,$$
$$q\colon \boldsymbol{R}^n \times \mathrm{SO}(n) \to \mathrm{SO}(n), \quad q(x, g) = g,$$

we observe that $q^{-1}\{g\} \cap \Phi = g(B) \times \{g\}$ and, for $0 \neq x \in \boldsymbol{R}^n$,

$$F_x = p^{-1}\{x\} \cap \Phi = \{(x, g) : g^{-1}(x) \in B\}$$

is a compact submanifold of Φ of dimension $m+l-n$ which is orientable since Φ is orientable.

Fix $o \in B \cap S^{n-1} = B \cap \{x : |x| = 1\}$. Define

$$\pi\colon \mathrm{SO}(n) \to S^{n-1}, \qquad \pi(g) = g(o).$$

Set $I = \mathrm{SO}(n) \cap \{g : g(o) = o\}$; I has dimension $\mu = \frac{1}{2}(n-1)(n-2)$. Assign a bi-invariant metric to $\mathrm{SO}(n)$ so that $\pi_{\#}(g)$ is an orthogonal projection for $g \in \mathrm{SO}(n)$. Orient $\mathrm{SO}(n)$ and denote by ω the unit positively oriented m form on $\mathrm{SO}(n)$. $e \in \mathrm{SO}(n)$ is the identity map of $\boldsymbol{R}^n$.

Denote

$$F = \mathrm{SO}(n) \cap \{g : g^{-1}(o) \in B\} = q(F_o) = \pi^{-1}(B)^{-1},$$

and observe that if $0 \neq x = |x|\, g_x(o)$, then

$$F_x = \{x\} \times g_x(F);$$

in particular, F_x is connected for $l>1$. Furthermore,

$$H^{m+l-n}(F_x) = H^{m+l-n}(F) = H^{\mu}(I)H^{l-1}(S^{l-1})$$
$$= \left[\prod_{i=2}^{n-1} i\alpha(i)\right] l\alpha(l).$$

Whenever $z \in \boldsymbol{R}^n$ define $\boldsymbol{r}_z(y)=|z-y|$ for each $y \in \boldsymbol{R}^n$; set $\boldsymbol{r}_0=\boldsymbol{r}$. We also define

$$\rho_z\colon \boldsymbol{R}^n \sim \{z\} \to S^{n-1}, \qquad \rho_z(x) = (x-z)/|x-z|;$$

set $\rho_0=\rho$.

$\boldsymbol{R}^n \times \mathrm{SO}(n)$ acts on $\boldsymbol{R}^n$ as the group of orientation preserving isometries of $\boldsymbol{R}^n$ through association of $(z, g) \in \boldsymbol{R}^n \times \mathrm{SO}(n)$ with the isometry, also denoted by (z, g), which maps x to $z+g(x)$. If $T \in \boldsymbol{E}_k(\boldsymbol{R}^n)$, we denote

$$(z, g)_{\#}T = z+g_{\#}T.$$

If $v \in \bigwedge_k(\boldsymbol{R}^n)$, $w \in \bigwedge_l(\boldsymbol{R}^n)$, $*v \wedge *w \neq 0$, $\omega \in \bigwedge^k(\boldsymbol{R}^n)$, $\zeta \in \bigwedge^l(\boldsymbol{R}^n)$, define

$$v \cap w = (-1)^t \|*v \wedge *w\|^{-1} * (*v \wedge *w) \in \textstyle\bigwedge_{k+l-n}(\boldsymbol{R}^n)$$

and

$$\omega \cap \zeta = (-1)^t * (*\omega \wedge *\zeta), \qquad t = (k+l)(n+1).$$

Recalling the integralgeometric constants $\gamma(n, k, l)$ and $\gamma^2(n, k, l)$ defined in [1, 2.13], we also define

$$\delta(n, k, l) = \gamma^2(n-1, k, l-1)(k+l-n)l\alpha(l)(2k+l-n)^{-1}.$$

Let X be an open subset of $\boldsymbol{R}^n$. $\boldsymbol{F}_k(X)$ is the closure of $\boldsymbol{N}_k(X)$ in $\boldsymbol{E}_k(X)$ with respect to the flat seminorm $\boldsymbol{F}$ [4, 4.1.12], which is complete relative to $\boldsymbol{F}$. Elements of $\boldsymbol{F}_k(X)$ are called k dimensional *flat chains*. Rectifiable currents are clearly flat; consequently, since the restriction of H^n to a bounded, open subset of $\boldsymbol{R}^n$ corresponds to a rectifiable n current, the restriction of H^n to a bounded Borel subset of $\boldsymbol{R}^n$ corresponds to a flat n current. Finally, we let

$$\boldsymbol{F}_k^{\mathrm{loc}}(X)$$

be the subset of $\boldsymbol{D}_k(X)$ consisting of all T such that

$$T \wedge \gamma \in \boldsymbol{F}_k(X) \quad \text{for every } \gamma \in \boldsymbol{D}^0(U).$$

Suppose $z \in \boldsymbol{R}^n$ and consider $T \in \boldsymbol{E}_k(\boldsymbol{R}^n)$. We shall denote by $T_z \in \boldsymbol{D}_k(\boldsymbol{R}^n \sim \{z\})$ the restriction of T to $\boldsymbol{D}^k(\boldsymbol{R}^n \sim \{z\})$. We recall from [4, 4.1.21] that if $T \in \boldsymbol{F}_k(\boldsymbol{R}^n)$ and $\boldsymbol{M}(T)<\infty$, then $\boldsymbol{M}(T_z)=\|T\|(\boldsymbol{R}^n \sim \{z\})=\boldsymbol{M}(T)$.

3. **Lifts of currents.** Let I act to the left on F by means of left translation. Since I is connected, this action will preserve an orientation of F. Since $F=\pi^{-1}(B)^{-1}$,

$$\boldsymbol{T}_e(F) = \pi_{\#}(e)^{-1}[\boldsymbol{T}_o(B \cap S^{n-1})];$$

thus, if $0 \neq w \in \bigwedge_{m+l-n} [\boldsymbol{T}_e(F)]$, and $v \in \bigwedge_{n-l} [\boldsymbol{T}_e(\mathrm{SO}(n))]$ is such that $\pi_\#(e)(v) = e_1 \wedge \cdots \wedge e_{n-l}$, then $v \wedge w \neq 0$. Choose w so that $\langle v \wedge w, \omega(e)\rangle > 0$, and for $l > 1$ orient F so that w is positively oriented. If $l = 1$, then F has two components. Let $\sigma \in \mathrm{SO}(n)$ be such that $\sigma(e_j) = e_j$ for $j < n-1$, and $\sigma(e_j) = -e_j$ for $j = n-1, n$. Right translation by σ leaves F invariant and permutes the components of F; orient F so that w is positively oriented, and so that right translation by σ reverses orientation.

3.1. LEMMA. *Φ is the bundle space of a fibre bundle $\mathscr{B}$ with fibre F, structure group I, projection $p|\Phi$ and base space $\boldsymbol{R}^n \sim \{0\}$.*

Proof. Let U be an open subset of $\boldsymbol{R}^n \sim \{0\}$ such that on $U_1 = \{x/|x| : x \in U\}$ there is a cross-section $\sigma_U \colon U_1 \to \mathrm{SO}(n)$ for π. Define a coordinate function $\varphi_U \colon U \times F \to \Phi$ by

$$\varphi_U(x, g) = (x, \sigma_U(x/|x|)g);$$

it is easy to verify that the set of φ_U provides Φ with the desired bundle structure.

3.2. REMARKS. One proceeds as in [1, §3] to define the *lifting map*

$$L_{\mathscr{B}} \colon \boldsymbol{E}_*(\boldsymbol{R}^n \sim \{0\}) \cap \{T : \boldsymbol{M}(T) < \infty\} \to \boldsymbol{E}_*(\Phi).$$

Inasmuch as $H^{m+l-n}(F) \neq 1$, suitable modifications must be made in the assertions of [1, §3] concerning the properties of $L_{\mathscr{B}}$.

Furthermore, [1, 3.5] can be used to show that if U is an open subset of $\boldsymbol{R}^n \sim \{0\}$ having compact closure, then there exists c such that $\boldsymbol{F}[L_{\mathscr{B}}(T)] \leqq c\boldsymbol{F}(T)$ whenever $T \in \boldsymbol{E}_*(U)$ and $\boldsymbol{M}(T) < \infty$. One uses this and the $\boldsymbol{F}$ completeness of $\boldsymbol{F}_*(\Phi)$ to extend $L_{\mathscr{B}}$ to an $\boldsymbol{F}$ continuous linear map of $\boldsymbol{F}_*(\boldsymbol{R}^n \sim \{0\})$ into $\boldsymbol{F}_*(\Phi)$; it is clear that [1, 3.3] remains valid. Moreover, since ∂ is $\boldsymbol{F}$ continuous, it follows from [1, 3.5(4)] that $\partial \circ L_{\mathscr{B}} = L_{\mathscr{B}} \circ \partial$.

Whenever $\gamma \in \mathrm{SO}(n)$ we define

$$\gamma_\Phi \colon \Phi \to \boldsymbol{R}^n \times \mathrm{SO}(n), \qquad \gamma_\Phi(x, g) = (\gamma(x), \gamma g).$$

γ_Φ is an isometry of Φ onto Φ, and it is easy to verify that γ_Φ induces a bundle map of $\mathscr{B}$ onto $\mathscr{B}$, hence we conclude from [1, 3.3 and 3.5(2)] that if $T \in \boldsymbol{E}_k(\boldsymbol{R}^n \sim \{0\})$ and $\boldsymbol{M}(T) < \infty$, then

$$\gamma_{\Phi\#}[L_{\mathscr{B}}(T)] = L_{\mathscr{B}}[\gamma_\#(T)], \qquad \gamma_{\Phi\#}[L_{\mathscr{B}}(T)^{\rightarrow}] = L_{\mathscr{B}}[\gamma_\#(T)]^{\rightarrow}.$$

In case $l = 1$ we define

$$\Sigma \colon \Phi \to \boldsymbol{R}^n \times \mathrm{SO}(n), \qquad \Sigma(x, g) = (x, g\sigma).$$

Σ is an isometry of Φ onto Φ. Suppose $T \in \boldsymbol{E}_k(\boldsymbol{R}^n \sim \{0\})$, $\boldsymbol{M}(T) < \infty$, and $\operatorname{spt} T$ lies in a coordinate neighborhood U with associated coordinate function $\varphi_U \colon U \times F \to \Phi$ defined as in the proof of 3.1. Observing that $\Sigma \circ \varphi_U = \varphi_U \circ (e \times \sigma)$

and $L_{\mathscr{B}}(T)=\varphi_U(T\times F)$ by [1, 3.3], we use [1, 3.3] in conjunction with a suitable partition of unity to conclude that for T with arbitrary support,

$$\Sigma_\#[L_{\mathscr{B}}(T)] = -L_{\mathscr{B}}(T), \qquad \Sigma_\#[L_{\mathscr{B}}(T)]^{\to} = -L_{\mathscr{B}}(T)^{\to}.$$

3.3 DEFINITION. For $(x, g) \in \Phi$, $\iota(x, g)$ is the linear right inverse of $p|\Phi_\#(x, g)$ whose range is orthogonal to $\boldsymbol{T}_{(x,g)}(F_x)$.

Let v be a k vectorfield on $\boldsymbol{R}^n$. ν^v is the function on Φ such that

$$\nu^v(x, g) = \|\iota(x, g)[v(x)]\|^{-1}.$$

3.4. LEMMA. *Suppose* $T\in \boldsymbol{E}_k(\boldsymbol{R}^n\sim\{0\})$, $\boldsymbol{M}(T)<\infty$.

(i) *If h is a bounded Baire function on* Φ, *then*

$$\|L_{\mathscr{B}}(T)\|(h\nu^{T^{\to}}) = \int_{\boldsymbol{R}^n}\int_{F_x} h(x, g)dH^{m+l-n}(x, g)d\|T\|x.$$

(ii) *If ψ is a bounded Baire $k+l-n$ form on* Φ, *then*

$$(-1)^t[L_{\mathscr{B}}(T)\wedge q^\#\omega](\psi)$$
$$= \int\langle(*[*T^{\to}(x) \wedge g_\#*B^{\to}], 0), \psi(x, g)\rangle\nu^{T^{\to}}(x, g)|x|^{l-n}d\|L_{\mathscr{B}}(T)\|(x, g),$$

where $t=(k+l-n)(m+l+1)$.

Proof. Using [1, 3.6(2)] and a suitable partition of unity we see that we can assume spt T to lie in a coordinate neighborhood U with associated coordinate function φ_U: $U\times F\to \Phi$ defined as in the proof of 3.1. From [1, 3.5(5)] we infer that, for $\|L_{\mathscr{B}}(T)\|$ almost all $(x, g) \in \Phi$,

$$L_{\mathscr{B}}(T)^{\to}(x, g) = \nu^{T^{\to}}(x, g)\iota(x, g)[T^{\to}(x)] \wedge F_x^{\to}(x, g). \tag{1}$$

(i) Denoting the right member of our conclusion by $\lambda(h)$, we see that $\lambda= \varphi_{U\#}(\|T\| \times \|F\|)$. On the other hand, (1) implies that

$$L_{\mathscr{B}}(T)^{\to}(x, g) = [\nu^{T^{\to}}\varphi_{U\#}(T^{\to}, 0) \wedge F_x^{\to}](x, g)$$

for $\|L_{\mathscr{B}}(T)\|$ almost all $(x, g) \in \Phi$, hence by [1, 3.3 and 3.1],

$$\nu^{T^{\to}}L_{\mathscr{B}}(T) = \nu^{T^{\to}}\varphi_{U\#}(T\times F) = [\nu^{T^{\to}}\varphi_{U\#}(T^{\to}, 0) \wedge F_x^{\to}]\lambda = L_{\mathscr{B}}(T)^{\to}\lambda.$$

(ii) We must show that for $\|L_{\mathscr{B}}(T)\|$ almost all $(x, g) \in \Phi$,

$$\begin{aligned}(-1)^t\langle L_{\mathscr{B}}(T)^{\to}, q^\#\omega \wedge \psi\rangle(x, g)& \\ = \langle(*[*T^{\to}(x) \wedge g_\#*B^{\to}], 0), \psi(x, g)\rangle\nu^{T^{\to}}(x, g)|x|^{l-n}.&\end{aligned} \tag{2}$$

From [1, 3.5] it follows that for $\|L_{\mathscr{B}}(T)\|$ almost all $(x, g) \in \Phi$,

$$\|L_{\mathscr{B}}(T)^{\to}(x, g)\| = \|T^{\to}(x)\| = 1, \tag{3}$$

hence we can use the maps γ_Φ (and Σ for $l=1$) which were introduced in 3.2 to infer that we need only verify (2) for (x, g) such that $x=|x|o$, $g_\#(B)=B$, and (1) and (3) hold.

We compute $i=\iota(x, g)$ as follows: Choose the cross-section σ_U used in defining φ_U so that

$$\sigma_{U\#}(o) = \text{transpose } \pi_{\#}(e).$$

Define I on SO (n) by $I(\gamma)=\gamma^{-1}$. Since $F_x=\{x\}\times F=\{x\}\times \pi^{-1}(B)^{-1}$, we have

$$\begin{aligned} T_{(x,g)}(F_x) &= \{0\} \oplus I_{\#}(g^{-1})[T_{g^{-1}}[\pi^{-1}(B)]] \\ &= \{0\} \oplus I_{\#}(g^{-1}) \circ L_{g^{-1}\#}(e)[T_e[\pi^{-1}(B)]] \\ &= \{0\} \oplus R_{g\#}(e)[T_e[\pi^{-1}(B)]]. \end{aligned}$$

On the other hand, it is easy to verify that for $1 \leqq j \leqq n-l$,

$$v_j = (e_j, |x|^{-1}R_{g\#}(e)\sigma_{U\#}(o)(e_j)) = \varphi_{U\#}(x, g)(e_j, 0);$$

v_j is clearly orthogonal to $T_{(x,g)}(F_x)$, hence

$$i(e_j) = v_j, \qquad 1 \leqq j \leqq n-l.$$

Finally, since $T_{(x,g)}(q^{-1}\{g\} \cap \Phi)=(B \cup \{0\}) \oplus \{0\}$,

$$i(e_j) = (e_j, 0) \quad \text{for } n-l+1 \leqq j \leqq n.$$

We next observe that

$$\alpha = T^{\to}(x)-(-1)^{k(n+1)}{*B^{\to}} \wedge *[*T^{\to}(x) \wedge *B^{\to}]$$

has the property $*\alpha \wedge *B^{\to}=0$, hence use (1) to conclude

$$\begin{aligned} &\langle L_{\mathscr{B}}(T)^{\to}(x, g), q^{\#}\omega \wedge \psi(x, g)\rangle \\ &\quad = \nu^{T^{\to}}(x, g)\langle[(-1)^{k(n+1)}i(*B^{\to}) \wedge (*[*T^{\to}(x) \wedge *B^{\to}], 0)+i(\alpha)] \wedge F_x^{\to}(x, g), \\ &\hspace{28em} q^{\#}\omega \wedge \psi(x, g)\rangle \\ &\quad = (-1)^{k(n+1)+(k+l-n)(m+l-n)}\nu^{T^{\to}}(x, g)\langle i(*B^{\to}) \wedge F_x^{\to}(x, g), q^{\#}\omega(x, g)\rangle \\ &\hspace{22em} \times\langle(*[*T^{\to}(x) \wedge *B^{\to}], 0), \psi(x, g)\rangle. \end{aligned}$$

Finally,

$$\begin{aligned} &\langle i(*B^{\to}) \wedge F_x^{\to}(x, g), q^{\#}\omega(x, g)\rangle \\ &\qquad = \langle q_{\#}(x, g)[i(*B^{\to})] \wedge F^{\to}(g), \omega(g)\rangle \\ &\qquad = (-1)^{l(n+1)}|x|^{l-n}\langle R_{g\#}(e)\sigma_{U\#}(o)(e_1 \wedge \cdots \wedge e_{n-l}) \wedge F^{\to}(g), \omega(g)\rangle \\ &\qquad = (-1)^{l(n+1)}|x|^{l-n}\langle \sigma_{U\#}(o)(e_1 \wedge \cdots \wedge e_{n-l}) \wedge F^{\to}(e), \omega(e)\rangle \\ &\qquad = (-1)^{l(n+1)}|x|^{l-n}, \end{aligned}$$

whence follows (2).

3.5. DEFINITIONS. Orient $R^n \times$ SO (n) so that $p^{\#}(dx^1 \wedge \cdots \wedge dx^n) \wedge q^{\#}\omega$ is positively oriented.

Let $\Omega=(0, \text{SO }(n)^{\to})$.

Let P: $\mathrm{SO}(n)\times B\to B$ be the projection. Referring to the basis of $\boldsymbol{T}_{(x,g)}(\Phi)$ constructed in the proof of 3.4 for $x=|x|o$, $g_\# B=B$, one infers that the intersection of $\boldsymbol{T}_{(x,g)}(\Phi)$ with the orthogonal complement $V_{(x,g)}$ of $\boldsymbol{T}_{(x,g)}[g(B)\times\{g\}]$ in $\boldsymbol{R}^n\oplus\{0\}$ is $\{0\}$. It follows from 3.2 that this remains true for arbitrary $(x,g)\in\Phi$. There therefore exists

$$\Upsilon\in \boldsymbol{E}^l[(\boldsymbol{R}^n\sim\{0\})\times \mathrm{SO}(n)]$$

such that

$$f^\#(\Upsilon)=P^\#(\psi_B)$$

and, for $(x,g)\in\Phi$,

$$\langle v\wedge w,\Upsilon(x,g)\rangle=0$$

whenever $v\in V_{(x,g)}$ and $w\in\bigwedge_{l-1}(\boldsymbol{T}_{(x,g)}[\boldsymbol{R}^n\times \mathrm{SO}(n)])$.

3.6. Lemma. *Let φ be a k form on $\boldsymbol{R}^n$ and $(x,g)\in\Phi$. If*

$$v\in\bigwedge_{k+l-n}(\boldsymbol{T}_{(x,g)}[g(B)\times\{g\}]),$$

then

$$(-1)^{m(n-l)}\langle v,p^\#[*(*\varphi\wedge g^{-1\#}*\psi_B)](x,g)\rangle$$
$$=\langle v\wedge\Omega(x,g),*[(p^\#*\varphi)\wedge*(\Upsilon\wedge q^\#\omega)](x,g)\rangle.$$

Proof. One uses [1, 4.2(2)] to verify that the left member of our assertion is equal to

$$(-1)^{m(n-l)}\langle v\wedge\Omega(x,g),*p^\#(*\varphi\wedge g^{-1\#}*\psi_B)(x,g)\rangle.$$

In order to complete the proof we need only compute

$$s=\langle w,(p^\# g^{-1\#}*\psi_B)(x,g)\rangle$$

for $w\in\bigwedge_{n-l}[\boldsymbol{T}_{(x,g)}(\boldsymbol{R}^n\times\{g\})]$. Denote by i the transpose of $p_\#(x,g)$ and by w_0 the orthogonal projection of $i[*p_\#(x,g)(w)]$ on $\boldsymbol{T}_{(x,g)}[g(B)\times\{g\}]$. Then, identifying $g(B)\times\{g\}$ with $g(B)$ and setting $y=g^{-1}(x)$, we have

$$f_\#(g,y)(0,g_\#^{-1}(w_0))=w_0$$

and by [1, 4.2(1)],

$$\begin{aligned}(-1)^{l(n-l)}s&=\langle g_\#^{-1}*p_\#(x,g)(w),\psi_B\rangle\\&=\langle g_\#^{-1}(w_0),\psi_B\rangle\\&=\langle P_\#(g,y)f_\#(g,y)^{-1}(w_0),\psi_B\rangle\\&=\langle w_0,\Upsilon(x,g)\rangle\\&=\langle i[*p_\#(x,g)(w)],\Upsilon(x,g)\rangle\\&=\langle i[*p_\#(x,g)(w)]\wedge\Omega(x,g),(\Upsilon\wedge q^\#\omega)(x,g)\rangle\\&=\langle *w,(\Upsilon\wedge q^\#\omega)(x,g)\rangle\\&=(-1)^{(l+m)(n-l)}\langle w,*(\Upsilon\wedge q^\#\omega)(x,g)\rangle.\end{aligned}$$

4. **Plane intersections of currents.** If $S \in \boldsymbol{F}_j(\Phi)$ and $j \geqq m$, then for H^m almost all $g \in \mathrm{SO}(n)$,

$$\langle S, q|\Phi, g\rangle \in \boldsymbol{F}_{j-m}(\Phi)$$

is the *slice of S by $q|\Phi$ over g* as characterized in [4, §4.3].

4.1. Definition. Suppose $T \in \boldsymbol{F}_k(\boldsymbol{R}^n \sim \{z\})$. Whenever

$$\langle L_{\mathscr{B}}(-z+T), q|\Phi, g\rangle \in \boldsymbol{F}_{k+l-n}(\Phi)$$

we define

$$T \cap (z+g_{\#}B) = (-1)^t[z+p_{\#}\langle L_{\mathscr{B}}(-z+T), q|\Phi, g\rangle],$$

where $t=(k+l-n)(m+l+1)+(k+l)(n+1)$.

4.2. Theorem. (1) *If $T \cap (z+g_{\#}B)$ exists, then*

$$\operatorname{spt}[T \cap (z+g_{\#}B)] \subset (\operatorname{spt} T) \cap (z+gB).$$

(2) *If U is an open subset of $\boldsymbol{R}^n \sim \{z\}$ having compact closure in $\boldsymbol{R}^n \sim \{z\}$, then there exists $c<\infty$ such that whenever $T \in \boldsymbol{F}_k(U)$,*

$$\int_{\mathrm{SO}(n)} \boldsymbol{F}[T \cap (z+g_{\#}B)]dH^m g \leqq c\boldsymbol{F}(T).$$

(3) *If $\boldsymbol{M}(T)<\infty$, then for H^m almost all $g \in \mathrm{SO}(n)$ the following are true*:

(i) $\boldsymbol{M}[T \cap (z+g_{\#}B)]<\infty$.

(ii) *For $\|T \cap (z+g_{\#}B)\|$ almost all $x \in \boldsymbol{R}^n$,*

$$[T \cap (z+g_{\#}B)]^{\rightarrow}(x) = T^{\rightarrow}(x) \cap g_{\#}B^{\rightarrow}.$$

(iii) *If T is an oriented proper submanifold of class* 1, *then $T \cap (z+gB)=A \cup Z$ where A is an orientable, proper $k+l-n$ submanifold of class* 1 *and $H^{k+l-n}(Z)=0$; if A is oriented according to* (ii), *then $A=T \cap (z+g_{\#}B)$.*

(4) *Whenever $\varphi \in \boldsymbol{E}^i(\boldsymbol{R}^n)$, $i \leqq k+l-n$, and $T \cap (z+g_{\#}B)$ exists, then so does*

$$(T \wedge \varphi) \cap (z+g_{\#}B) = [T \cap (z+g_{\#}B)] \wedge \varphi.$$

(5) *If $k+l>n$ and $T \cap (z+g_{\#}B)$ exists, then*

$$\partial[T \cap (z+g_{\#}B)] = (-1)^{n-l}\partial T \cap (z+g_{\#}B).$$

Furthermore, if T is normal, then $T \cap (z+g_{\#}B)$ is normal for H^m almost all $g \in \mathrm{SO}(n)$.

Proof. (1) follows from [1, 3.5(3)] and [4, 4.3.1].

(2) follows from 3.2, [4, 4.3.1 and 4.3.2(5)].

(3)(i) follows from [1, 3.5(2)] and [4, 4.3.2(2)].

(3)(ii) follows from 3.4(ii) and [4, 4.3.2(1) and (2)].

(3)(iii) follows from the coarea formula [4, 3.2.12], [1, 3.5(6)], [4, 4.3.8] and (ii).

(4) follows from [4, 4.3.1] and the second statement of [1, 3.6(2)].

(5) follows from 3.2, [4, 4.3.1] and (3)(i).

4.3. REMARK. 4.2(2) and (3) and [4, 4.1.23] show that the intersections $T \cap (z+g_{\#}B)$ are intrinsically determined by the action of SO (n) on $\boldsymbol{R}^n$. In particular, if $w, z \in \boldsymbol{R}^n$ and $h \in \mathrm{SO}(n)$, then

$$(w+h_{\#})[T \cap (z+g_{\#}B)] = (w+h_{\#}T) \cap [w+h(z)+hg_{\#}B]$$

for H^m almost all $g \in \mathrm{SO}(n)$.

That our intersections are usually the same as those defined in [1, §4] will follow in §5.

Suppose $z \in \boldsymbol{R}^n$ and consider $T \in \boldsymbol{F}_k(\boldsymbol{R}^n)$. Let $\mathscr{U}$ be a partition of unity subordinate to a locally finite open cover of $\boldsymbol{R}^n \sim \{z\}$. Then for H^m almost all $g \in \mathrm{SO}(n)$,

$$(T \wedge u) \cap (z+g_{\#}B) \in \boldsymbol{F}_{k+l-n}(\boldsymbol{R}^n)$$

for each $u \in \mathscr{U}$; for such g we define

$$T \cap (z+g_{\#}B) = \sum_{u \in \mathscr{U}} (T \wedge u) \cap (z+g_{\#}B) \in \boldsymbol{F}^{\mathrm{loc}}_{k+l-n}(\boldsymbol{R}^n \sim \{z\}).$$

One uses 4.2(4) to verify that this definition does not depend on the choice of $\mathscr{U}$, and that if $z \notin \operatorname{spt} T$, then the definition reduces to the one in 4.1. With appropriate modifications the assertions of 4.2 hold for T.

4.4. DEFINITION. If $T \in \boldsymbol{F}_k(\boldsymbol{R}^n)$ and $z \in \boldsymbol{R}^n$, then

$$V^T(z) = H^{\mu}(I)^{-1} \int_{\mathrm{SO}(n)} \boldsymbol{M}[T \cap (z+g_{\#}B)]\, dH^m g.$$

If $k=n-1$, $l=1$ and S corresponds to the restriction of H^n to a bounded Borel set C, then $\frac{1}{2}V^{\partial S}$ coincides with the function v^C_∞ introduced in [6].

4.5. REMARK. If T is a rectifiable k current, then it follows from [5, 8.16], 4.2(3)(iii) and [1, 5.8 and 5.9] that for $r>0$ and $T_{z,r}=T \cap \{x : |x-z|<r\}$,

$$\int_{\boldsymbol{R}^n} V^{T_{z,r}}(z)\, dH^n z = \gamma(n, k, l)\alpha(l) r^l \boldsymbol{M}(T).$$

If S is the n current corresponding to the restriction of H^n to a bounded open set C, and if $\boldsymbol{M}(\partial S)=\boldsymbol{P}(C)<\infty$, then S is rectifiable, hence ∂S is rectifiable by [5, 8.14]. If $l=1$, then $\frac{1}{2}V^{(\partial S)_{z,r}}(z)$ is equal to $v^C_r(z)$ as defined in [6], whence follows the integralgeometric formula mentioned in §1.

5. Integralgeometric formulas.

Whenever $0 \neq x \in \boldsymbol{R}^n$ define

$$I_x = \mathrm{SO}(n) \cap \{g : g(x) = x\}$$

and choose g_x so that $x=|x|\, g_x(o)$. Since

$$I_x = g_x I g_x^{-1}, \qquad H^{\mu}(I_x) = H^{\mu}(I) = H^{n-1}(S^{n-1})^{-1} H^m[\mathrm{SO}(n)].$$

5.1. LEMMA. *Suppose $T \in \boldsymbol{E}_k(\boldsymbol{R}^n)$, $\boldsymbol{M}(T) < \infty$, φ is a bounded Baire k form with support in $\boldsymbol{R}^n \sim \{0\}$, and h is the bounded Baire function on Φ such that*

$$h(x, g) = \langle *\vec{T}(x) \wedge g_{\#}*\vec{B}, *\varphi(x) \wedge g^{-1\#}*\psi_B \rangle.$$

Then

$$\begin{aligned} H^{\mu}(I)^{-1} \int_{\boldsymbol{R}^n} \int_{I_x g_x} & h(x, g)\, dH^{\mu} g\, d\|T\| x \\ &= \gamma^2(n-1, k, l-1) T(\varphi) \quad \textit{for } \varphi = f_0 \rho^{\#} \varphi_0, \\ &= \gamma^2(n-1, k-1, l-1) T(\varphi) \quad \textit{for } \varphi = (f_1 \rho^{\#} \varphi_1) \wedge d\boldsymbol{r}, k+l > n. \end{aligned}$$

Proof. Suppose $\varphi = f_0 \rho^{\#} \varphi_0$ and $\|\vec{T}(x)\| = 1$. Observing that we can assume $\vec{T}(x)$ to lie in the orthogonal complement X of $\boldsymbol{R}x$, we apply [1, 6.2, 9.3 and 2.13] to obtain

$$H^{\mu}(I)^{-1} \int_{I_x} h(x, g g_x)\, dH^{\mu} g = \gamma^2(n-1, k, l-1) \langle \vec{T}(x), \varphi(x) \rangle.$$

Now suppose $\varphi = (f_1 \rho^{\#} \varphi_1) \wedge d\boldsymbol{r}$ and $\|\vec{T}(x)\| = 1$. Observing that we can assume $\vec{T}(x) = u \wedge x$, where u lies in X, we again apply [1, 6.2] in X to obtain our conclusion.

5.2. LEMMA. *Suppose h is a bounded Baire function on Φ such that if $g_0 \in \mathrm{SO}(n)$ and $g_{0\#} B = B$, then $h(x, g g_0) = h(x, g)$ for $(x, g) \in \Phi$. Then*

$$l\alpha(l) \int_{I_x g_x} h(x, g)\, dH^{\mu} g = \int_{F_x} h(x, g)\, dH^{m+l-n}(x, g).$$

Proof. First observe that

$$\begin{aligned} H^{\mu}(I) \int_{F_x} h(x, \gamma)\, dH^{m+l-n}(x, \gamma) &= \int_{I_x} \int_{F_x} h(x, g\gamma)\, dH^{m+l-n}(x, \gamma)\, dH^{\mu} g. \\ &= \int_{F_x} \int_{I_x} h(x, g\gamma)\, dH^{\mu} g\, dH^{m+l-n}(x, \gamma). \end{aligned}$$

Fix $(x, \gamma) \in F_x$ and denote $\gamma^{-1}(x) = y \in B$. Choose g_0 so that $g_0(y) = |y| o$, $g_{0\#} B = B$, and set $g_x' = \gamma g_0^{-1}$. Then for some $g_1 \in I_x$ we have $g_x = g_1^{-1} g_x'$ and

$$\begin{aligned} \int_{I_x} h(x, g\gamma)\, dH^{\mu} g &= \int_{I_x} h(x, g g_x' g_0)\, dH^{\mu} g \\ &= \int_{I_x} h(x, g g_x')\, dH^{\mu} g = \int_{I_x g_x} h(x, g)\, dH^{\mu} g. \end{aligned}$$

5.3. THEOREM. *If $T \in \boldsymbol{F}_k(\boldsymbol{R}^n)$ and $\varphi \in \boldsymbol{E}_k(\boldsymbol{R}^n \sim \{z\})$, then*

$$\begin{aligned} H^{\mu}(I)^{-1} \int_{\mathrm{SO}(n)} & T \cap (z + g_{\#} B)(\boldsymbol{r}_z^{n-l} \varphi \cap g^{-1\#} \psi_B)\, dH^m g \\ &= l\alpha(l) \gamma^2(n-1, k, l-1) T(\varphi) \qquad \textit{for } \varphi = f_0 \rho_z^{\#} \varphi_0 \\ &= l\alpha(l) \gamma^2(n-1, k-1, l-1) T(\varphi) \quad \textit{for } \varphi = (f_1 \rho_z^{\#} \varphi_1) \wedge d\boldsymbol{r}_z, k+l > n. \end{aligned}$$

Proof. With the assistance of 4.2(2) we see that we can assume that $z=0$, $0 \notin \operatorname{spt} T$, and $\boldsymbol{M}(T)<\infty$. Referring to 3.5 we let Ξ be the $k+l-n$ form on $\boldsymbol{R}^n \times \mathrm{SO}(n)$ such that whenever

$$v \in \bigwedge_{k+l-n} [\boldsymbol{T}_{(x,g)}(\boldsymbol{R}^n \times \mathrm{SO}(n))],$$

$$(-1)^{m(n-l)}\langle v, \Xi(x,g)\rangle = \langle v \wedge \Omega(x,g), *[(p^{\#}*\varphi) \wedge *(\Psi \wedge q^{\#}\omega)](x,g)\rangle.$$

Suppose $\varphi = f_0\rho^{\#}\varphi_0$. We apply 5.1, 5.2, 3.4(i), 3.6, 3.4(ii), [4, 4.3.2(1)], 3.6 and [1, 2.5] to verify that

$$\begin{aligned}
&H^{m+l-n}(F)\gamma^2(n-1,k,l-1)T(\varphi)\\
&\quad= \int_{\Phi} \langle (*[*\vec{T}(x) \wedge g_{\#}*\vec{B}], 0), \Xi(x,g)\rangle \nu^{\vec{T}}(x,g)\, d\|L_{\mathscr{B}}(T)\|(x,g)\\
&\quad= (-1)^t L_{\mathscr{B}}(T) \wedge q^{\#}\omega(\boldsymbol{r}^{n-l} \circ p\Xi)\\
&\quad= (-1)^t \int_{\mathrm{SO}(n)} \langle L_{\mathscr{B}}(T), q, g\rangle(\boldsymbol{r}^{n-l} \circ p\Xi)\, dH^m g\\
&\quad= (-1)^{(k+l)(n+1)} \int_{\mathrm{SO}(n)} T \cap g_{\#}B[\boldsymbol{r}^{n-l}*(*\varphi \wedge g^{-1\#}*\psi_B)]\, dH^m g\\
&\quad= \int_{\mathrm{SO}(n)} T \cap g_{\#}B(\boldsymbol{r}^{n-l}\varphi \cap g^{-1\#}\psi_B)\, dH^m g.
\end{aligned}$$

The proof for $\varphi = (f_1\rho^{\#}\varphi_1) \wedge d\boldsymbol{r}$ is analogous.

5.4. LEMMA. (1) *If $z \in \boldsymbol{R}^n$, then*

$$\Sigma_z = \rho_z^{\#}[\boldsymbol{E}^k(S^{n-1})] \cup \rho_z^{\#}[\boldsymbol{E}^{k-1}(S^{n-1})] \wedge d\boldsymbol{r}_z$$

spans $\boldsymbol{E}^k(\boldsymbol{R}^n \sim \{z\})$ over $\boldsymbol{E}^0(\boldsymbol{R}^n \sim \{z\})$.

(2) *If $z_1, \ldots, z_{k+1} \in \boldsymbol{R}^n$ determine a k-plane Π, then*

$$\Sigma = \bigcup_{i=1}^{k+1} \rho_{z_i}^{\#}[\boldsymbol{E}^k(S^{n-1})]$$

spans $\boldsymbol{E}^k(\boldsymbol{R}^n \sim \Pi)$ over $\boldsymbol{E}^0(\boldsymbol{R}^n \sim \Pi)$.

Proof. (1) It is clear that for each $z \neq x \in \boldsymbol{R}^n$, $\Sigma_z(x)$ spans $\bigwedge^k(\boldsymbol{R}^n)$. Thus choose $\psi_i \in \Sigma_z$, $i=1,\ldots,\nu=\binom{n}{k}$, such that $\psi_1(x),\ldots,\psi_\nu(x)$ is a basis of $\bigwedge^k(\boldsymbol{R}^n)$. Then $\psi_1(y),\ldots,\psi_\nu(y)$ are linearly independent for y in some neighborhood N of x, hence $\psi_1|N,\ldots,\psi_\nu|N$ span $\boldsymbol{E}^k(N)$ over $\boldsymbol{E}^0(N)$, and our conclusion follows with use of a suitable partition of unity.

(2) Fix $x \in \boldsymbol{R}^n \sim \Pi$. Then $\bigwedge^k(\boldsymbol{R}^n)$ is spanned by $\Sigma(x)$. In fact, if this were not true there would exist $0 \neq w \in \bigwedge_k(\boldsymbol{R}^n)$ such that for $i=1,\ldots,k+1$,

$$\langle w, \rho_{z_i\#}[\boldsymbol{E}^k(S^{n-1})](x)\rangle = \{0\},$$

or equivalently, $\rho_{z_i\#}(x)(w)=0$. But for each i,

$$\dim\ker\left[\rho_{z_i\#}(x) \mid \bigwedge_k(\boldsymbol{R}^n)\right] = \binom{n-1}{k-1},$$

hence we would have $w=(x-z_i)\wedge w_i$, and the linear independence of the $(x-z_i)$, $i=1,\dots,k+1$, would allow us to conclude that

$$w = \alpha(x-z_1) \wedge \cdots \wedge (x-z_k) \notin \ker \rho_{z_{k+1}\#}(x).$$

Proceeding as in (1) we conclude that for some neighborhood N of x, $\Sigma|N$ spans $\boldsymbol{E}^k(N)$ over $\boldsymbol{E}^0(N)$, and our conclusion follows with use of a suitable partition of unity.

5.5. Definition. If $z\in\boldsymbol{R}^n$, then $\boldsymbol{D}^k(z)$ is the linear subspace of $\boldsymbol{D}^k(\boldsymbol{R}^n\sim\{z\})$ consisting of sums of the form $\sum_{j=1}^N f_j\rho_z^{\#}\varphi_j$ with $f_j\in\boldsymbol{D}^0(\boldsymbol{R}^n\sim\{z\})=\boldsymbol{D}^0(z)$ and $\varphi_j\in\boldsymbol{D}^k(S^{n-1})$.

5.6. Lemma. *If $T\in\boldsymbol{F}_k(\boldsymbol{R}^n)$ and $\boldsymbol{M}(T)<\infty$, then*

$$H^\mu(I)V^T(0) = l\alpha(l)\int_{\boldsymbol{R}^n\sim\{0\}}\int_{I_xg_x} \|{*T^{\rightarrow}}(x)\wedge g_{\#}{*B^{\rightarrow}}\|\,|x|^{l-n}\,dH^\mu g\,d\|T\|x.$$

Proof. Choosing $f_i\in\boldsymbol{D}^0(0)$, $i=1,2,\dots$, such that $0\leqq f_1\leqq f_2\leqq\cdots$ and

$$\lim_{i\to\infty} f_i(y) = 1 \quad \text{for } y\neq 0,$$

we apply 4.4, 4.1, [4, 4.3.2(2)], 3.4(ii), 3.4(i) and 5.2 to obtain

$$\begin{aligned} H^\mu(I)V^{T\wedge f_i}(0) &= \boldsymbol{M}[L_{\mathscr{B}}(T\wedge f_i)\wedge q^{\#}\omega] \\ &= \int_\Phi \|{*T^{\rightarrow}}(x)\wedge g_{\#}{*B^{\rightarrow}}\|\nu^{T^{\rightarrow}}(x,g)|x|^{l-n}\,d\|L_{\mathscr{B}}(T\wedge f_i)\|(x,g) \\ &= l\alpha(l)\int_{\boldsymbol{R}^n}\int_{I_xg_x}\|{*T^{\rightarrow}}(x)\wedge g_{\#}{*B^{\rightarrow}}\|\,dH^\mu g|x|^{l-n}f_i(x)\,d\|T\|x. \end{aligned}$$

Consequently, we infer from 4.2(4) that

$$\begin{aligned} l\alpha(l)\int_{\boldsymbol{R}^n\sim\{0\}}\int_{I_xg_x} \|{*T^{\rightarrow}}(x)\wedge g_{\#}{*B^{\rightarrow}}\|\,dH^\mu g|x|^{l-n}\,d\|T\|x & \\ &= H^\mu(I)\lim_{i\to\infty} V^{T\wedge f_i}(0) \\ &\geqq \int_{\mathbf{SO}(n)}\liminf_{i\to\infty}\boldsymbol{M}[(T\cap g_{\#}B)\wedge f_i]\,dH^m g \\ &\geqq \int_{\mathbf{SO}(n)}\boldsymbol{M}(T\cap g_{\#}B)\,dH^m g \\ &= H^\mu(I)V^T(0), \end{aligned}$$

which implies our assertion for the case $V^T(0)=\infty$. On the other hand, if $V^T(0)<\infty$, then

$$\begin{aligned} H^\mu(I)V^{T\wedge f_i}(0) &= \int_{\mathbf{SO}(n)}\boldsymbol{M}[(T\cap g_{\#}B)\wedge f_i]\,dH^m g \\ &= \int_{\mathbf{SO}(n)}\|T\cap g_{\#}B\|(f_i)\,dH^m g, \end{aligned}$$

hence

$$\lim_{i\to\infty} V^{T\wedge f_i}(0) = V^T(0),$$

which implies our assertion for such T.

5.7. LEMMA. *Suppose $r+s \leqq n$, $v \in \bigwedge_r(\mathbf{R}^n)$, and $w \in \bigwedge_s(\mathbf{R}^n)$.*

(1) *If $r \neq n/2$ or $s \neq n/2$, then*

$$H^m[\mathrm{SO}(n)]\gamma^2(n, n-r, n-s)\|v\| \, \|w\| \leqq \binom{r+s}{r} \int_{\mathrm{SO}(n)} \|v \wedge g_\# w\| \, dH^m g.$$

If v or w is simple, then this is true for arbitrary r and s, and the factor $\binom{r+s}{r}$ can be omitted.

(2) $$\int_{\mathrm{SO}(n)} \|v \wedge g_\# w\| \, dH^m g \leqq H^m[\mathrm{SO}(n)]\gamma(n, n-r, n-s)\|v\| \, \|w\|.$$

If v and w are simple, then equality holds. If v is not simple and the smallest linear subspace L of $\mathbf{R}^n$ containing v has dimension $\lambda \leqq r+s+1$, or if an analogous condition holds for w, then the inequality is strict.

Proof. (1) Consider $\varphi \in \bigwedge^r(\mathbf{R}^n)$ and $\psi \in \bigwedge^s(\mathbf{R}^n)$ such that $\|\varphi\| \leqq 1$ and $\|\psi\| \leqq 1$. Then by [1, 6.2 and 9.3]

$$H^m[\mathrm{SO}(n)]\gamma^2(n, n-r, n-s)\langle v, \varphi\rangle\langle w, \psi\rangle \leqq \binom{r+s}{r} \int_{\mathrm{SO}(n)} \|v \wedge g_\# w\| \, dH^m g$$

if $r \neq n/2$ or $s \neq n/2$, whence follows the first statement. If w is simple, let ψ be the metric dual of $w/|w|$. Then $\langle w, \psi\rangle = |w|$ and

$$H^m[\mathrm{SO}(n)]\gamma^2(n, n-r, n-s)\langle v, \varphi\rangle|w| \leqq \int_{\mathrm{SO}(n)} \|v \wedge g_\# w\| \, dH^m g.$$

(2) If v and w are simple, equality follows from [1, 5.6 and 5.9]. On the other hand, suppose v is not simple and use [5, 2.2] to obtain simple r vectors $\alpha_1, \ldots, \alpha_M$ and simple s vectors $\beta_1, \ldots, \beta_N$ such that α_1 and α_2 are linearly independent,

$$v = \sum_{i=1}^{M} \alpha_i, \qquad \|v\| = \sum_{i=1}^{M} |\alpha_i|,$$

and

$$w = \sum_{i=1}^{N} \beta_i, \qquad \|w\| = \sum_{i=1}^{N} |\beta_i|.$$

Inasmuch as

$$\begin{aligned}\int_{\mathrm{SO}(n)} \|v \wedge g_\# w\| \, dH^m g &\leqq \sum_{j=1}^{N} \sum_{i=1}^{M} \int_{\mathrm{SO}(n)} |\alpha_i \wedge g_\# \beta_j| \, dH^m g \\ &= H^m[\mathrm{SO}(n)]\gamma(n, n-r, n-s) \sum_{j=1}^{N} \sum_{i=1}^{M} |\alpha_i| \, |\beta_j| \\ &= H^m[\mathrm{SO}(n)]\gamma(n, n-r, n-s)\|v\| \, \|w\|,\end{aligned}$$

the remainder of (2) will follow if we can find a simple s vector $\beta \neq 0$ such that

(*) $$\|(\alpha_1+\alpha_2) \wedge \beta\| < |\alpha_1 \wedge \beta| + |\alpha_2 \wedge \beta|.$$

First assume $r+s=n$. Corresponding to $j=1, 2$ let P_j be the subset of

$$W = \bigwedge_s (\boldsymbol{R}^n) \cap \{\beta : \beta \text{ is simple and } |\beta| = 1\}$$

consisting of β for which $\alpha_j \wedge \beta$ is positively oriented, $Q_j = -P_j$, and $R_j = W \sim (P_j \cup Q_j)$. Then P_j and Q_j are open in W, R_j has no interior, and R_j is the boundary of P_j and of Q_j. Further, $R_1 \neq R_2$. In fact, if A_j is the r dimensional linear subspace of $\boldsymbol{R}^n$ containing α_j, C_j is the orthogonal complement of $A_1 \cap A_2$ in A_j, and D is the orthogonal complement of $A_1 + A_2$ in $\boldsymbol{R}^n$, then

$$\dim (D+C_j) = s.$$

If $0 \neq \beta_0 \in \bigwedge_s (D+C_1)$, then

$$\alpha_1 \wedge \beta_0 = 0, \qquad \alpha_2 \wedge \beta_0 \neq 0.$$

Therefore, $P_1 \cap Q_2 \neq \varnothing$; any $\beta \in P_1 \cap Q_2$ will satisfy (*).

In case $\lambda \leqq r+s$ we apply the result of the last paragraph with $\boldsymbol{R}^n$ replaced by L to obtain a simple $\beta_0 \in \bigwedge_{\lambda - r}(L)$ such that

$$\|(\alpha_1+\alpha_2) \wedge \beta_0\| = |(\alpha_1+\alpha_2) \wedge \beta_0| < |\alpha_1 \wedge \beta_0| + |\alpha_2 \wedge \beta_0|.$$

Thus if $\beta_1 \neq 0$ is a simple $r+s-\lambda$ vector lying in the orthogonal complement of L, $\beta = \beta_0 \wedge \beta_1$ will satisfy (*).

In case $\lambda = r+s+1$ we choose β_0 as above, choose a simple s vector β and $\beta_2 \in L$ such that $\beta_0 = \beta \wedge \beta_2$, and conclude that

$$\|(\alpha_1+\alpha_2) \wedge \beta\| = |(\alpha_1+\alpha_2) \wedge \beta| < |\alpha_1 \wedge \beta| + |\alpha_2 \wedge \beta|.$$

5.8. Example. Let $n=6$, $r=3$, $s=1$, and

$$v = \tfrac{1}{2}[e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6].$$

One uses a procedure similar to the one used on [7, p. 54] to show that

$$\|v\| = \tfrac{1}{2}|e_1 \wedge e_2 \wedge e_3| + \tfrac{1}{2}|e_4 \wedge e_5 \wedge e_6| = 1.$$

Let w_1 be a linear combination of e_1, e_2, e_3, and w_2 be a linear combination of e_4, e_5, e_6 such that $|w_1| = |w_2| = 1$. Then if $w = \alpha w_1 + \beta w_2$, one uses the fact that $\|e_1 \wedge e_2 + e_3 \wedge e_4\| = 2$ to verify that

$$\|v \wedge w\| = \tfrac{1}{2}(|\alpha| + |\beta|) = \tfrac{1}{2}|e_1 \wedge e_2 \wedge e_3 \wedge w| + \tfrac{1}{2}|e_4 \wedge e_5 \wedge e_6 \wedge w|;$$

consequently,

$$\int_{\mathrm{SO}(6)} \|v \wedge g_{\#} w_1\| \, dH^{15} g = H^{15}[\mathrm{SO}(6)]\gamma(6, 3, 5).$$

5.9. Theorem. *Assume $T \in \boldsymbol{F}_k(\boldsymbol{R}^n)$ and $z \in \boldsymbol{R}^n$.*

$$\text{(1)} \qquad \delta(n, k, l)\boldsymbol{M}(T_z \wedge \boldsymbol{r}_z^{l-n}) \leqq V^T(z) \leqq l\alpha(l)\boldsymbol{M}(T_z \wedge \boldsymbol{r}_z^{l-n}).$$

Consequently,

$$V^T(z) \leqq l\alpha(l)\boldsymbol{M}(T_z)[\operatorname{dist}(z, \operatorname{spt} T)]^{l-n},$$
$$\delta(n, k, l)\boldsymbol{M}(T_z) \leqq V^T(z) \sup\{|x-z| : x \in \operatorname{spt} T\}^{n-l}.$$

(2) *Suppose $r>0$ and T lies in the image of $\boldsymbol{E}_k(z+rS^{n-1})$ under the map induced by the inclusion of $z+rS^{n-1}$ in $\boldsymbol{R}^n$. Then*

$$l\alpha(l)\gamma^2(n-1, k, l-1)r^{l-n}\boldsymbol{M}(T) \leqq V^T(z) \leqq l\alpha(l)\gamma(n-1, k, l-1)r^{l-n}\boldsymbol{M}(T).$$

Suppose $\boldsymbol{M}(T)<\infty$. The second inequality becomes equality when $\overrightarrow{T}$ is simple. If for x in a set of positive $\|T\|$ measure it is true that $\overrightarrow{T}(x)$ is not simple and the smallest linear subspace of $\boldsymbol{R}^n$ containing $*\overrightarrow{T}(x)$ has dimension not greater than $2n-k-l+1$, then the second inequality is strict.

(3) *Suppose $l=n-k$ and T is rectifiable with k dimensional density Θ^k. Then*

$$V^T(z) = (n-k)\alpha(n-k)\gamma(n-1, k, n-k-1) \int_{S^{n-1}} \sum_{\rho_z^{-1}\{\theta\}} \Theta^k \, dH^k\theta.$$

(4) *Suppose $l=n-k$ and $\boldsymbol{M}(T)<\infty$. Whenever $z \neq x \in \boldsymbol{R}^n$ and $\|\overrightarrow{T}(x)\|=1$ let $\tau(x)$ be the orthogonal projection of $\overrightarrow{T}(x)$* on *$\boldsymbol{T}_x(z+|x-z|S^{n-1})$. Define*

$$\gamma_z(x) = H^\mu(I)^{-1}\|\tau(x)\|^{-1} \int_{I_{x-z}} |\overrightarrow{T}(x) \wedge gg_{x-z\#}\overrightarrow{B}| \, dH^\mu g$$

whenever $\|\overrightarrow{T}(x)\|=1$. Then

$$\begin{aligned}
&(n-k)\alpha(n-k)\gamma^2(n-1, k, n-k-1) \sup\{T(\boldsymbol{r}_z^{-k}\varphi) : \varphi \in \boldsymbol{D}^k(z), \boldsymbol{M}(\varphi) \leqq 1\} \\
&\quad\leqq V^T(z) = (n-k)\alpha(n-k) \sup\{T(\gamma_z\boldsymbol{r}_z^{-k}\varphi) : \varphi \in \boldsymbol{D}^k(z), \boldsymbol{M}(\varphi) \leqq 1\} \\
&\quad\leqq (n-k)\alpha(n-k)\gamma(n-1, k, n-k-1) \sup\{T(\boldsymbol{r}_z^{-k}\varphi) : \varphi \in \boldsymbol{D}^k(z), \boldsymbol{M}(\varphi) \leqq 1\}.
\end{aligned}$$

The second inequality becomes equality if $\rho_{z\#}[\overrightarrow{T}(x)]$ is simple for $\|T\|$ almost all $x \in \boldsymbol{R}^n \sim \{z\}$, in which case

$$\gamma_z = \gamma(n-1, k, n-k-1).$$

(5) *If $l=n-k$, $\boldsymbol{M}(T)<\infty$, and $\gamma_z(x)$ is a continuous function of z for $\|T\|$ almost all x, then V^T is lower semicontinuous.*

Proof. We can assume $z=0$. Choose $f_i \in \boldsymbol{D}^0(0)$, $i=1, 2, \ldots$, such that $0 \leqq f_1 \leqq f_2 \leqq \cdots \leqq 1$ and whenever K is a compact subset of $\boldsymbol{R}^n \sim \{0\}$ there exists i for which $f_i|K=1$.

If $\boldsymbol{M}(T)<\infty$, then the second inequality of (1) follows from 5.6. In general, we use 4.2(4) to infer that

$$\begin{aligned}
V^T(0) &\leqq \liminf_{i\to\infty} V^{T \wedge f_i}(0) \\
&\leqq l\alpha(l) \liminf_{i\to\infty} \boldsymbol{M}(T_z \wedge f_i \wedge \boldsymbol{r}_z^{l-n}) \\
&\leqq l\alpha(l)\boldsymbol{M}(T_z \wedge \boldsymbol{r}_z^{l-n}).
\end{aligned}$$

Since $\delta(n, k, n-k)=0$, we assume $k+l>n$. Fix $\varphi \in \boldsymbol{E}^k(\boldsymbol{R}^n \sim \{0\})$ such that $\boldsymbol{M}(\varphi) \leqq 1$, and use 5.4 to write $\varphi=\varphi_1+\varphi_2$ where $\varphi_1 \in \boldsymbol{D}^k(0)$ and $\varphi_2 \in \boldsymbol{D}^{k-1}(0) \wedge d\boldsymbol{r}$. Noting that $\boldsymbol{M}(\varphi_1) \leqq 1$ and $\boldsymbol{M}(\varphi_2) \leqq 1$, we obtain from 5.3

$$\begin{aligned} l\alpha(l)H^\mu(I)T(\boldsymbol{r}^{l-n}\varphi) &= \gamma^2(n-1, k, l-1)^{-1} \int T \cap g_\# B(\varphi_1 \cap g^{-1\#}\psi_B)\, dH^m g \\ &\quad + \gamma^2(n-1, k-1, l-1)^{-1} \int T \cap g_\# B(\varphi_2 \cap g^{-1\#}\psi_B)\, dH^m g \\ &\leqq l\alpha(l)\delta(n, k, l)^{-1} H^\mu(I) V^T(0). \end{aligned}$$

(2) follows from 5.6 together with 5.7 with $\boldsymbol{R}^n$ replaced by $\boldsymbol{T}_x(rS^{n-1})$.

Suppose $l=n-k$. Consider $0 \neq x \in \boldsymbol{R}^n$. Inasmuch as

$$T^{\rightarrow}(x) \wedge g_\# B^{\rightarrow} = \tau(x) \wedge g_\# B^{\rightarrow}$$

for $g \in I_x g_x$, 5.7 implies that

$$(*) \qquad \gamma^2(n-1, k, n-k-1) \leqq \gamma_0(x) \leqq \gamma(n-1, k, n-k-1),$$

with the second inequality becoming equality if $\tau(x)$ is simple. Moreover,

$$|x|^{-k}\tau(x) = \rho_\#(x)[T^{\rightarrow}(x)],$$

and if T is rectifiable we infer from 5.6 and [5, 8.16] that

$$(**) \qquad \begin{aligned} V^T(0) &= (n-k)\alpha(n-k)\gamma(n-1, k, n-k-1) \int_{\boldsymbol{R}^n \sim \{0\}} |x|^{-k} |\tau(x)|\, d\|T\| x \\ &= (n-k)\alpha(n-k)\gamma(n-1, k, n-k-1) \int_A |\rho_\#(x)[T^{\rightarrow}(x)]|\, \Theta^k(x)\, dH^k x, \end{aligned}$$

where A is the union of a countable family of proper k dimensional submanifolds of class 1 of $\boldsymbol{R}^n$ and $T^{\rightarrow}(x)$ is tangent to one of these submanifolds for $\|T\|$ almost all x. (3) now follows from application of the area formula [4, 3.2.3] to $\rho | A$.

We next observe that

$$\begin{aligned} &\sup \{T(\boldsymbol{r}^{-k}\varphi) : \varphi \in \boldsymbol{D}^k(0), \boldsymbol{M}(\varphi) \leqq 1\} \\ &\qquad = \sup \left\{ \int \boldsymbol{r}^{-k} \langle \tau, \varphi \rangle \, d\|T\| : \varphi \in \boldsymbol{D}^k(0), \boldsymbol{M}(\varphi) \leqq 1 \right\} \\ &\qquad = \sup \left\{ \int \boldsymbol{r}^{-k} \langle \tau, \psi \rangle \, d\|T\| : \psi \in \boldsymbol{D}^k(\boldsymbol{R}^n \sim \{0\}), \boldsymbol{M}(\psi) \leqq 1 \right\} \\ &\qquad = \int \boldsymbol{r}^{-k} \|\tau\| \, d\|T\|, \end{aligned}$$

and similarly,

$$\int \boldsymbol{r}^{-k}\gamma_0 \|\tau\|\, d\|T\| = \sup \{T(\gamma_0 \boldsymbol{r}^{-k}\varphi) : \varphi \in \boldsymbol{D}^k(0), \boldsymbol{M}(\varphi) \leqq 1\}.$$

(4) now follows from 5.6 and (*).

In order to prove (5) it suffices to prove lower semicontinuity at 0. We can assume $V^T(0)>0$. Fix $0<\varepsilon<V^T(0)$ and choose $\varphi\in \boldsymbol{D}^k(0)$ with

$$\varepsilon < (n-k)\alpha(n-k)T(\gamma_0\boldsymbol{r}^{-k}\varphi) \leqq V^T(0).$$

Inasmuch as $\varphi_y=(-y, e)^{\#}\varphi\in \boldsymbol{D}^k(y)$ for $y\in \boldsymbol{R}^n$,

$$\lim_{y\to 0}\gamma_y(x)\boldsymbol{r}_y^{-k}(x)\varphi_y(x) = \gamma_0(x)\boldsymbol{r}^{-k}(x)\varphi(x)$$

for $\|T\|$ almost all $x\in \boldsymbol{R}^n$, and $\|\gamma_y\boldsymbol{r}_y^{-k}\varphi_y\|\leqq \boldsymbol{M}(\boldsymbol{r}^{-k}\varphi)$, we can use Lebesgue's dominated convergence theorem to conclude that

$$\begin{aligned}\liminf_{y\to 0} V^T(y) &\geqq \liminf_{y\to 0}(n-k)\alpha(n-k)T(\gamma_y\boldsymbol{r}_y^{-k}\varphi_y)\\ &= (n-k)\alpha(n-k)T(\gamma_0\boldsymbol{r}^{-k}\varphi) > \varepsilon,\end{aligned}$$

whence

$$\liminf_{y\to 0} V^T(y) \geqq V^T(0).$$

5.10. Remark. Suppose S is the restriction of H^n to a bounded Borel set C and φ_0 is the unit $n-1$ form on S^{n-1} such that $d\boldsymbol{r}\wedge\varphi_0$ is positively oriented. Then for $h\in \boldsymbol{D}^0(z)$, $\partial S(h\rho_z^{\#}\varphi_0)$ is the double layer potential $W_h(z)$ considered by Král in [6]. In this case 5.3 (with $T=\partial S$) reduces to the formula derived in [6, 2.5]. Furthermore, if $\boldsymbol{M}(\partial S)=\boldsymbol{P}(C)<\infty$, then 5.9(4) reduces to the formula given for v_∞^C in [6, 1.6].

5.11. Corollary. *Consider $T\in \boldsymbol{F}_k(\boldsymbol{R}^n)$.*

(1) *If there exist $z_1,\dots,z_{k+2}\in \boldsymbol{R}^n$ which do not lie on a k-plane, such that for $i=1,\dots,k+2$,*

$$(\operatorname{spt} T)\cap(z_i+gB) = \varnothing$$

for H^m almost all $g\in \mathrm{SO}(n)$, then

$$T = 0.$$

(2) *If $k+l>n$ and there exists $z\in \boldsymbol{R}^n$ such that for H^m almost all $g\in\mathrm{SO}(n)$,*

$$(\operatorname{spt} T)\cap(z+gB)$$

has measure 0 with respect to $k+l-n$ dimensional integralgeometric measure in $z+g(B)$ (in particular, if this set is $\varnothing$), then $T=0$.

Proof. Consider (1) and assume $l=n-k$. By 4.2(1), $V^T(z_i)=0$ for $i=1,\dots,k+2$. Let Π_j denote the k-plane containing $\{z_k : k\neq j\}$, and fix $\varphi\in \boldsymbol{D}^k(\boldsymbol{R}^n\sim\Pi_j)$. Recalling 5.4(2) and applying 5.3 we conclude that $T(\varphi)=0$. Inasmuch as

$$\bigcup\{\boldsymbol{R}^n\sim\Pi_j : j=1,\dots,k+2\} = \boldsymbol{R}^n,$$

the proof is complete.

Consider (2). From [4, 4.1.20] it follows that $T\cap(z+g_{\#}B)=0$ for H^m almost all $g\in\mathrm{SO}(n)$, hence $V^T(z)=0$, and therefore $T=0$ by 5.9(1).

5.12. COROLLARY. *Suppose* $T \in \boldsymbol{F}_k(\boldsymbol{R}^n)$, $A \subset \boldsymbol{R}^n$, *and the hypothesis of either* 5.10(1) *or* (2) *is satisfied with* "spt T" *replaced by* "(spt T) $\cap A$".

(1) *If A is open, then* spt $T \subset \boldsymbol{R}^n \sim A$.

(2) *If A is a Borel set and* $\boldsymbol{M}(T) < \infty$, *then* $\|T\|(A) = 0$.

Proof. One argues as in the proof of [4, 4.1.21], using 5.11 in place of [4, 4.1.20].

5.13. REMARK. Theorem 6.6 in [1] is erroneous. However, the following is true:

Let X, G and Ψ be as in [1, §6]. *Consider* $S \in \boldsymbol{N}_k(X)$ *and* $T \in \boldsymbol{N}_l(X)$.

(1) *If $k > n/2$ or $l > n/2$, then*

$$\binom{2n-k-l}{n-k}^{-1} \gamma^2(n, k, l)\boldsymbol{M}(S)\boldsymbol{M}(T) \leqq \int_G \boldsymbol{M}(S \cap g_\# T)\, d\Psi g.$$

If $\vec{S}$ or $\vec{T}$ is simple, then this is true for arbitrary k and l, and the factor $\binom{2n-k-l}{n-k}^{-1}$ *can be omitted*

$$(2) \qquad \int_G \boldsymbol{M}(S \cap g_\# T)\, d\Psi g \leqq \gamma(n, k, l)\boldsymbol{M}(S)\boldsymbol{M}(T).$$

If $\vec{S}$ and $\vec{T}$ are simple, then equality holds. If for x in a set of positive $\|S\|$ measure, $\vec{S}(x)$ is not simple and the smallest linear subspace of $\boldsymbol{T}_x(X)$ containing $*\vec{S}(x)$ has dimension not greater than $2n-k-l+1$, or if an analogous condition holds for $\vec{T}$, then the inequality is strict.

These statements follow from use of 5.7 in conjunction with [1, 5.5].

5.14. REMARK. We will now examine the relationship between the intersections defined in 4.1 and those defined in [1, §4].

First note that the slicing theory in [4, §4.3] can be used to extend the intersection theory in [1, §4] to flat chains having finite mass.

Fix $T \in \boldsymbol{F}_k(\boldsymbol{R}^n)$, $\boldsymbol{M}(T) < \infty$, let U be a compact neighborhood of spt T, and use [4, 4.1.23] to obtain polyhedral chains $T_1, T_2, \ldots$ lying in U such that $\boldsymbol{M}(T_i) \leqq 2\boldsymbol{M}(T)$ and $T_i \to T$ as $i \to \infty$. Fix $u_1 \in \boldsymbol{D}^0(\boldsymbol{R}^n \sim U)$, $\boldsymbol{M}(u_1) \leqq 1$, $u_2 \in \boldsymbol{D}^0[\mathrm{SO}(n)]$, $\boldsymbol{M}(u_2) \leqq 1$, and $\psi \in \boldsymbol{D}^{k+l-n}(\boldsymbol{R}^n)$. Denoting intersection in the sense of [1] by $\cap_1$, we use [1, 4.4(4) and 4.9] to conclude that whenever $R > \operatorname{diam}(U \cup \operatorname{spt} u_1)$,

$$T_i \cap_1 (z + g_\# B) = T_i \cap_1 [z + g_\#(B \cap \{x : |x| < R\})]$$

for $H^n \times H^m$ almost all $(z, g) \in (\operatorname{spt} u_1) \times \mathrm{SO}(n)$, hence by [1, 4.4(3)]

$$T \cap_1 (z + g_\# B) = T \cap_1 [z + g_\#(B \cap \{x : |x| < R\})]$$

for $H^n \times H^m$ almost all $(z, g) \in (\operatorname{spt} u_1) \times \mathrm{SO}(n)$. Therefore, by [1, 4.4(2)]

$$\mathscr{I}_1 = \int_{\boldsymbol{R}^n \times \mathrm{SO}(n)} T \cap_1 (z + g_\# B)(\psi) u_1(z) u_2(g)\, dH^n \times H^m(z, g) < \infty,$$

and Fubini's theorem applies to give

$$\mathscr{I}_1 = \int_{\boldsymbol{R}^n} \int_{\mathrm{SO}(n)} T \cap_1 (z + g_\# B)(\psi) u_1(z) u_2(g)\, dH^m g\, dH^n z.$$

By [1, 4.4(4)] and 4.2(3) we have for H^n almost all $z \in \boldsymbol{R}^n \sim U$,

$$(-1)^c T_i \cap (z+g_\# B) = T_i \cap_1 (z+g_\# B)$$

for H^m almost all $g \in \mathrm{SO}(n)$, where $c=n(k+l)+kl$. We use [1, 4.4(3), 4.4(2)] and Fubini's theorem to infer that

$$(-1)^c \mathscr{I}_1 = \lim_{i\to\infty} \int_{\boldsymbol{R}^n} \int_{\mathrm{SO}(n)} T_i \cap (z+g_\# B)(\psi) u_1(z) u_2(g)\, dH^m g\, dH^n z.$$

Now by 5.9(1) each of the inner integrals is not greater in absolute value than

$$l\alpha(l) 2\boldsymbol{M}(T)\boldsymbol{M}(\psi) \operatorname{dist}(\operatorname{spt} u_1, U)^{l-n} |u_1(z)| H^m[\mathrm{SO}(n)],$$

hence by 4.2(2) we can apply Lebesgue's dominated convergence theorem to obtain

$$(-1)^c \mathscr{I}_1 = \int_{\boldsymbol{R}^n} \int_{\mathrm{SO}(n)} T \cap (z+g_\# B)(\psi) u_1(z) u_2(g)\, dH^m g\, dH^n z.$$

We therefore conclude that *for H^n almost all $z \notin \operatorname{spt} T$,*

$$(-1)^c T \cap (z+g_\# B) = T \cap_1 (z+g_\# B)$$

for H^m almost all $g \in \mathrm{SO}(n)$.

Next suppose T lies in the image of $\boldsymbol{E}_k(S^{n-1})$ under the map induced by the inclusion of S^{n-1} in $\boldsymbol{R}^n$. Orient S^{n-1} so that for $x \in S^{n-1}$, $x \wedge (S^{n-1})^{\to}(x)$ is positively oriented in $\boldsymbol{R}^n$. Denote

$$B' = B \cap S^{n-1};$$

orient B' so that for $x \in B'$, $x \wedge B'^{\to}$ is positively oriented in B. *Then for H^m almost all $g \in \mathrm{SO}(n)$,*

$$(-1)^c T \cap g_\# B = -T \cap g_\# B'$$

as defined in [1, §4] *with* $X=S^{n-1}$, $G=\mathrm{SO}(n)$.

In order to show this, we infer from [4, 4.1.23], [1, 4.4(3) and (4)], and 4.2(2) and (3) that it is only necessary to show that for $x \in B'$ and $v \in \bigwedge_k [T_x(S^{n-1})]$,

$$(-1)^c v \cap B^{\to} = -v \cap_0 B'^{\to},$$

where $\cap_0$ refers to use of the star operator in $\bigwedge_* [T_x(S^{n-1})]$. One verifies this directly from the definitions of the star operators.

6. Boundaries of flat chains of finite mass.

6.1. Lemma. *Suppose $S \in \boldsymbol{F}_{k+1}(\boldsymbol{R}^n)$, $\boldsymbol{M}(S) < \infty$, and T, $U \in \boldsymbol{E}_k(\boldsymbol{R}^n)$ with*

$$\partial S = T + U, \qquad \boldsymbol{M}(T) < \infty,$$

and for some $z \in \boldsymbol{R}^n$,

$$(\operatorname{spt} U) \cap (z+gB) = \varnothing$$

for H^m almost all $g \in \mathrm{SO}(n)$. Then

$$\gamma^2(n-1, k, l-1)\partial S(\varphi) \leqq \boldsymbol{M}(T)\boldsymbol{M}(\varphi) \quad \text{for } \varphi \in \boldsymbol{D}^k(z),$$
$$\gamma^2(n-1, k-1, l-1)\partial S(\varphi) \leqq \boldsymbol{M}(T)\boldsymbol{M}(\varphi) \quad \text{for } \varphi \in \boldsymbol{D}^{k-1}(z) \wedge d\boldsymbol{r}_z,\ k+l > n.$$

Proof. Our proof was suggested by the proof of [3, 2.2].

We can assume $z=0 \notin (\operatorname{spt} S) \cup (\operatorname{spt} T)$.

Using local coordinates for the bundle $\mathscr{B}$ in conjunction with a suitable partition of unity for $\boldsymbol{R}^n \sim \{0\}$, one applies [1, 3.3] to verify that

$$\operatorname{spt}[\partial L_{\mathscr{B}}(S) - L_{\mathscr{B}}(T)] \subset C = p^{-1}(\operatorname{spt} U) \cap \Phi.$$

Choose $h_1, h_2, \ldots \in \boldsymbol{E}^0(\mathrm{SO}(n))$ such that for $j=1, 2, \ldots$,

$$\operatorname{spt} h_j \subset \mathrm{SO}(n) \sim q(C), \qquad 0 \leqq h_j \leqq 1,$$

and $\lim_{j\to\infty} h_j(g)=1$ for $g \in \mathrm{SO}(n) \sim q(C)$. Then for each $j=1, 2, \ldots$,

$$\partial[L_{\mathscr{B}}(S) \wedge q^{\#}(h_j\omega)] = \partial L_{\mathscr{B}}(S) \wedge q^{\#}(h_j\omega) = L_{\mathscr{B}}(T) \wedge q^{\#}(h_j\omega),$$

because $d(h_j\omega)=0$ and

$$\operatorname{spt} q^{\#}(h_j\omega) \cap \operatorname{spt}[\partial L_{\mathscr{B}}(S) - L_{\mathscr{B}}(T)] = \varnothing.$$

Now $C' = q^{-1}[q(C)]$ is closed and $H^m[q(C')]=0$, hence for H^m almost all $g \in \mathrm{SO}(n)$, $\|\langle L_{\mathscr{B}}(S), q \mid \Phi, g\rangle\|(C')=0$, and therefore by [4, 4.3.2(2)],

$$\|L_{\mathscr{B}}(S) \wedge q^{\#}\omega\|(C') = 0.$$

We apply Lebesgue's dominated convergence theorem and [1, 3.5(2)] to obtain for $\psi \in \boldsymbol{E}^{k+l-n}(\Phi)$,

$$\begin{aligned} L_{\mathscr{B}}(S)[(q^{\#}\omega) \wedge d\psi] &= \lim_{j\to\infty} L_{\mathscr{B}}(S)[(q^{\#}h_j\omega) \wedge d\psi] \\ &= \lim_{j\to\infty} L_{\mathscr{B}}(T)[(q^{\#}h_j\omega) \wedge \psi] \\ &= L_{\mathscr{B}}(T) \cap (\Phi \sim C')[(q^{\#}\omega) \wedge \psi]. \end{aligned}$$

Consider $\varphi \in \boldsymbol{D}^k(0)$ and define $\Xi \in \boldsymbol{D}^{k+l-n}(\Phi)$ as in the proof of 5.3. Then from [4, 4.3.2(1)], 3.6, 4.2(5) and 5.3 we have

$$\begin{aligned} (-1)^t\partial[L_{\mathscr{B}}(S) \wedge q^{\#}\omega](\boldsymbol{r}^{n-l} \circ p\Xi) &= (-1)^t \int_{\mathrm{SO}(n)} \partial\langle L_{\mathscr{B}}(S), q \mid \Phi, g\rangle(\boldsymbol{r}^{n-l} \circ p\Xi)\, dH^m g \\ &= \int_{\mathrm{SO}(n)} \partial S \cap g_{\#}B(\boldsymbol{r}^{n-l}\varphi \cap g^{-1\#}\psi_B)\, dH^m g \\ &= l\alpha(l)H^\mu(I)\gamma^2(n-1, k, l-1)\partial S(\varphi), \end{aligned}$$

where $t=(k+l-n)(m+l+1)+m$. Finally, we conclude using 3.4(ii), 3.4(i), 5.2, 3.6 and 5.1 that this is equal to

$$(-1)^t(L_{\mathscr{B}}(T) \wedge q^{\#}\omega) \cap (\Phi \sim C')(\boldsymbol{r}^{n-l} \circ p\Xi) \leqq l\alpha(l)H^\mu(I)\boldsymbol{M}(T)\boldsymbol{M}(\varphi).$$

Verification of the inequality for $\varphi \in \boldsymbol{D}^{k-1}(z) \wedge d\boldsymbol{r}_z$, $k+l>n$, is analogous.

6.2. THEOREM. *Assume* $S \in \boldsymbol{F}_{k+1}(\boldsymbol{R}^n)$, $\boldsymbol{M}(S)<\infty$, and $z \in \boldsymbol{R}^n$.

(1) $$\delta(n,k,l)\boldsymbol{M}(\partial S) \leqq 4\binom{n}{k} V^{\partial S}(z) \sup\{|x-z| : x \in \operatorname{spt} \partial S\}^{n-l}.$$

(2) *Under the hypothesis of* 6.1,

$$\delta(n,k,l)\boldsymbol{M}(\partial S) \leqq 4\binom{n}{k} l\alpha(l)\boldsymbol{M}(T).$$

(3) *Suppose* $\|S\|(\boldsymbol{r}_z^{-k})<\infty$, $h \in \boldsymbol{E}^0(\boldsymbol{R}^n)$, $\varphi \in \boldsymbol{E}^k(S^{n-1})$, $\boldsymbol{M}(h) \leqq 1$ *and* $\boldsymbol{M}(\varphi) \leqq 1$. *Then*

$$\gamma^2(n-1,k,n-k-1)S[d(h \wedge \rho_z^{\#}\varphi)]$$
$$\leqq 4V_{n-k}^{\partial S}(z)+\gamma^2(n-1,k+1,n-k-2)^{-1}h(z)\boldsymbol{M}(d\varphi)V_{n-k-1}^S(z).$$

(V_l *indicates use of* l *planes*; *the second term is* 0 *if* $k+1=n$.)

Proof. We can assume that $z=0$.

Consider $h \in \boldsymbol{E}^0(\boldsymbol{R}^n)$ such that $h(0)=0$. Corresponding to each $j=1,2,\ldots$ we choose $\varepsilon_j>0$, $\delta_j>0$ and $x_j \in \boldsymbol{R}^n$ such that

$$\lim_{j\to\infty} \varepsilon_j = \lim_{j\to\infty} \delta_j = 0, \qquad |h(x_j)| = \sup\{|h(x)| : |x| \leqq \delta_j\} = \varepsilon_j,$$

and $|x_j|=\delta_j$. Using the mean value theorem, we see that we can assume that $\lim_{j\to\infty} \varepsilon_j/\delta_j=|dh(0)|$ and $2\varepsilon_j \geqq |dh(0)|\delta_j$. There exists $\eta_j \in \boldsymbol{E}^0(\boldsymbol{R})$ such that

$$\eta_j(r) = 0 \quad \text{for } r \leqq \delta_j/j,$$
$$\eta_j(r) = (1-j^{-1})\varepsilon_j^{-1}\delta_j|dh(0)| \quad \text{for } r \geqq \delta_j,$$
$$0 \leqq \eta_j'(r) \leqq 2|dh(0)|/\varepsilon_j \quad \text{for } r \in \boldsymbol{R};$$

define $h_j=\eta_j \circ \boldsymbol{r}$. Then for $0 \neq x \in \boldsymbol{R}^n$,

$$0 \leqq h_j(x) \leqq 2, \qquad |dh_j(x)|\,|h(x)| \leqq 2|dh(0)|,$$
$$\lim_{j\to\infty} h_j(x) = 1, \qquad \lim_{j\to\infty} dh_j(x) = 0.$$

Recalling from [4, 4.1.21] that $\|S\|\{0\}=0$, we write $\varphi_0=h\,dx^{i_1}\wedge\cdots\wedge dx^{i_k}$,

$$\partial S(h_j\varphi_0) = S(dh_j \wedge \varphi_0)+S(h_j\,d\varphi_0),$$

and use Lebesgue's dominated convergence theorem to conclude that

$$\lim_{j\to\infty} \partial S(h_j\varphi_0) = \partial S(\varphi_0).$$

It follows that $\partial S(\varphi_0) \leqq 2\boldsymbol{M}(\varphi_0)\boldsymbol{M}[(\partial S)_z]$, hence for $\varphi \in \boldsymbol{E}^k(\boldsymbol{R}^n)$, $\varphi(0)=0$, $\boldsymbol{M}(\varphi) \leqq 2$,

$$\partial S(\varphi) \leqq 4\binom{n}{k}\boldsymbol{M}[(\partial S)_z].$$

We obtain this relation for arbitrary φ with $M(\varphi) \leqq 1$ by observing that $\partial S(\varphi) = \partial S[\varphi - \varphi(0)]$. (1) now follows from 5.9(1).

Since $\delta(n, k, n-k) = 0$, we assume $k + l > n$ for the proof of (2). Consider $\varphi \in D^k(R^n \sim \{0\})$, $M(\varphi) \leqq 1$. Using 5.4 to write $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1 \in D^k(0)$ and $\varphi_2 \in D^{k-1}(0) \wedge dr$, we apply 6.1 to φ_1 and to φ_2 to obtain $\delta(n, k, l)\partial S(\varphi) \leqq l\alpha(l)M(T)$. Consequently, $\delta(n, k, l)M[(\partial S)_z] \leqq l\alpha(l)M(T)$, whence follows (2).

Turning to (3), we choose h_j for $h_0 = h - h(0)$ and infer from 5.3 that since $\|\rho^\# \varphi\| \leqq r^{-k}$,

$$(n-k)\alpha(n-k)\gamma^2(n-1, k, n-k-1)S[d(h_j h_0 \wedge \rho^\# \varphi)] \leqq 4V^{\partial S}_{n-k}(0)$$

for $j = 1, 2, \ldots$. As we found previously,

$$S[d(h_j h_0) \wedge \rho^\# \varphi] \to S(dh_0 \wedge \rho^\# \varphi) = S(dh \wedge \rho^\# \varphi) \quad \text{as } j \to \infty.$$

If $k+1 = n$, then $d\varphi = 0$ and the proof is complete. On the other hand, if $k + 1 < n$, then we use 5.7, 5.6 and [4, 4.1.21] and proceed as in the derivation of equation (**) in the proof of 5.9 to infer that

$$\begin{aligned}\int_{R^n} |\langle S^{\rightarrow}, \rho^\# d\varphi \rangle| \, d\|S\| &\leqq M(d\varphi) \int_{R^n} \|\rho_\#(x) S^{\rightarrow}(x)\| \, d\|S\| x \\ &\leqq [(n-k-1)\alpha(n-k-1)\gamma^2(n-1, k+1, n-k-2)]^{-1} M(d\varphi) V^S_{n-k-1}(0).\end{aligned}$$

Consequently, Lebesgue's dominated convergence theorem implies that if $V^S_{n-k-1}(0) < \infty$, then

$$S(h_j h_0 \wedge d\rho^\# \varphi) \to S(h_0 \wedge d\rho^\# \varphi) \quad \text{as } j \to \infty,$$

whence

$$(n-k)\alpha(n-k)\gamma^2(n-1, k, n-k-1)S[d(h \wedge \rho^\# \varphi)] \leqq 4V^{\partial S}_{n-k}(0) + |h(0)S(d\rho^\# \varphi)|,$$

and our assertion is clear.

6.3. THEOREM. *Suppose $S \in F_{k+1}(R^n)$, $M(S) < \infty$, and $z_1, \ldots, z_{k+2} \in R^n$ do not lie on a k-plane. If either $V^{\partial S}(z_i) < \infty$ or the hypothesis of* 6.1 *holds for z_i for each $i = 1, \ldots, k+2$, then*

$$S \in N_{k+1}(R^n).$$

Proof. The proof was suggested by the proof of [6, 2.10]. Let $\varphi = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. It suffices to show that

$$M(\partial S \wedge \varphi) < \infty \quad \text{for } 1 \leqq i_1 < \cdots < i_k \leqq n.$$

Let Π_j denote the k-plane containing $\{z_k : k \neq j\}$. Observing that

$$\{R^n \sim \Pi_j : j = 1, \ldots, k+2\}$$

covers R^n, we choose a subordinate partition of unity $\alpha_1, \ldots, \alpha_{k+2}$ with $\Pi_j \cap \operatorname{spt} \alpha_j = \varnothing$. Thus it is sufficient to show that

$$M(\partial S \wedge \alpha_j \varphi) < \infty \quad \text{for } j = 1, \ldots, k+2.$$

Consider, for instance, $j=k+2$. By 5.4 there exist $\beta_1, \dots, \beta_{k+1} \in \boldsymbol{E}^0(\boldsymbol{R}^n)$, $\operatorname{spt} \beta_i \subset \operatorname{spt} \alpha_{k+2}$, and $\varphi_i \in \rho^{\#}_{z_i}[\boldsymbol{E}^k(S^{n-1})]$ such that

$$\alpha_{k+2}\varphi = \sum_{i=1}^{k+1} \beta_i \varphi_i.$$

Thus it is sufficient to show that

$$\boldsymbol{M}(\partial S \wedge \beta_i \varphi_i) < \infty \quad \text{for } i = 1, \dots, k+1.$$

Fix $h \in \boldsymbol{E}^0(\boldsymbol{R}^n)$, $\boldsymbol{M}(h) \leqq 1$. Then from 5.3 and 4.2 we infer that

$$l\alpha(l)\gamma^2(n-1, k, l-1)\partial S(h\beta_i\varphi_i) \leqq V^{\partial S}(z_i)\boldsymbol{M}(\beta_i\varphi_i) \sup \{|x-z_i| : x \in \operatorname{spt} \partial S\}^{n-l}.$$

Moreover, if the hypothesis of 6.1 is satisfied, then

$$\gamma^2(n-1, k, l-1)\partial S(h\beta_i\varphi_i) \leqq \boldsymbol{M}(T)\boldsymbol{M}(\beta_i\varphi_i).$$

Therefore, $\boldsymbol{M}(\partial S \wedge \beta_i \varphi_i) < \infty$.

6.4. Remark. Let S be a rectifiable $k+1$ current. If one of the following conditions is satisfied, then $\boldsymbol{M}(\partial S) < \infty$, hence S is an integral current by [4, 4.2.16]:

(1) The hypothesis of 6.3.

(2) $k+l>n$ and for some $z \in \boldsymbol{R}^n$ either $V^{\partial S}(z) < \infty$ or the hypothesis of 6.1 is satisfied.

Conversely, if $\boldsymbol{M}(\partial S) < \infty$, then from 5.9(1) it follows that $V^{\partial S}(z) < \infty$ for $z \notin \operatorname{spt} \partial S$.

7. **V^T for T a manifold.** Let $T \in \boldsymbol{F}_k(\boldsymbol{R}^n)$, $\boldsymbol{M}(T) < \infty$, be obtained by integration over an oriented k dimensional proper submanifold of class 1, which is also denoted by T. Assume $z \in T$.

7.1. Theorem. $\|T\|(\boldsymbol{r}_z^{-k+1}) = \int_T \boldsymbol{r}_z^{-k+1}\, dH^k < \infty$. *Consequently, if* $k+l>n$, *then by* 5.9(1), $V^T(z) < \infty$.

Proof. This is easily verified by introducing a bi-Lipschitzian coordinate system for T in a neighborhood of z.

For the remainder of this section we assume $k+l=n$.

7.2. Lemma.

$$V^T(0) = (n-k)\alpha(n-k)\gamma(n-1, k, n-k-1) \int |\vec{T}(x) \wedge x|\, |x|^{-k-1}\, dH^k x.$$

Proof. Recalling equation (**) in the proof of 5.9(3), we observe that for $x \neq 0$,

$$|\rho_{\#}(x)[\vec{T}(x)]| = |\vec{T}(x) \wedge x|\, |x|^{-k-1}.$$

7.3. Example. Let $n=2$ and $k=1$. Whenever $|t| < \frac{1}{2}$ define

$$g(t) = \int_0^t \int_0^u -[ln|v|]^{-1}\, dH^1 v\, dH^1 u.$$

Let T be the graph of the function F defined by

$$F(t) = g(t)\sin(1/t),$$

oriented so that

$$T^{\rightarrow}(t, F(t)) = [1+F'(t)^2]^{-1/2}[e_1+F'(t)e_2].$$

Then T is of class 1, but not of class 2, and by 7.2

$$V^T(0) \geqq \int_{-\varepsilon}^{\varepsilon} \left| \frac{F'(t)}{t} - \frac{F(t)}{t^2} \right| dH^1 t = \infty$$

for sufficiently small positive numbers ε.

7.4. THEOREM. *Suppose there exist an open subset $U \subset \mathbf{R}^k$, a diffeomorphism $F: U \rightarrow T$ with $z \in F(U)$, and constants $A > 0$ and $0 < \alpha \leqq 1$ for which*

$$|F_\#(x)(v) - F_\#(y)(v)| < A|x-y|^\alpha$$

whenever $x, y \in U$ and $|v| \leqq 1$. Then $V^T(z) < \infty$.

Proof. We can assume that $z=0$, that $\mathbf{T}_0(T)$ is spanned by $e_1, \ldots, e_k$ and, by 5.9(1), that $T = F(U)$. Let $P: \mathbf{R}^n \rightarrow \mathbf{T}_0(T)$ be the orthogonal projection. It is easy to see that the condition on $F_\#$ holds (with a different A) for $(F \circ \varphi)_\#$ where φ is a Lipschitzian diffeomorphism of an open subset V of $\mathbf{R}^k$ onto U and

$$\sup\{|\varphi_\#(x)(v)| : x \in V,\ |v| \leqq 1\} < \infty.$$

Thus we can assume that $U = \mathbf{R}^k \cap \{x : |x| < \varepsilon\}$,

$$F^{-1} = P_0 = P|T$$

and the Jacobian $J_k P_0 > \frac{1}{2}$. Observe that

$$\sigma = \sup\{|F_\#(x)(v)| : x \in U \text{ and } |v| \leqq 1\} < \infty.$$

Then by 7.2 and the area formula [4, 3.2.3],

$$\begin{aligned}
&[(n-k)\alpha(n-k)]^{-1} V^T(0) \\
&\qquad \leqq 2 \int_U |T^{\rightarrow}[F(x)] \wedge F(x)|\, |F(x)|^{-2} |x|^{-k+1}\, dH^k x \\
&\qquad = 2 \int_{S^{k-1}} \int_0^{\varepsilon} |T^{\rightarrow}[F(t\theta)] \wedge F(t\theta)|\, |F(t\theta)|^{-2}\, dH^1 t\, dH^{k-1}\theta \\
&\qquad \leqq 2 \int_{S^{k-1}} \int_0^{\varepsilon} |F_\theta'(t) \wedge F_\theta(t)|\, |F_\theta(t)|^{-2}\, dH^1 t\, dH^{k-1}\theta,
\end{aligned}$$

where $F_\theta(t) = F(t\theta)$. Thus fix θ and observe that in evaluating the inner integral we can assume that

$$F_\theta(t) = te_1 + \varphi(t),$$

where $\varphi(t)$ involves only $e_{k+1}, \ldots, e_n$ and $\varphi'(0)=0$. It is a routine matter to verify with the help of the mean value theorem that for $0<t<\varepsilon$,

$$\begin{aligned} |F_\theta'(t) \wedge F_\theta(t)|\,|F_\theta(t)|^{-2} &\leqq (1+\sigma)t^{-2}|\varphi(t)|+At^{\alpha-1} \\ &\leqq [A+(1+\sigma)(n-k)A]t^{\alpha-1}, \end{aligned}$$

whence $V^T(0)<\infty$.

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